Lagrangian Cobordism Functor in Sheaf Theory

Uppsala Symplectic Seminar

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Northwestern

Legendrians and Lagrangian cobordisms

Definition

A contact manifold (Y, ξ) is a $(2n+1)$-dimensional manifold with a maximally non-integrable hyperplane distribution ξ. A Legendrian submanifold $\Lambda \subset (Y, \xi)$ is an n-dimensional submanifold such that $T\Lambda \subset \xi$.

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Definition

Let $\Lambda_-, \Lambda_+ \subset (Y, \ker \alpha)$ be Legendrian submanifolds. An exact Lagrangian cobordism L from Λ_- to Λ_+ is an exact Lagrangian $L \subset (\mathbb{R}_t \times Y, d(e^t \alpha))$, such that

- for $T \gg 0$, $L \cap (-\infty, -T) \times Y = (-\infty, -T) \times \Lambda_-$;
- for $T \gg 0$, $L \cap (T, +\infty) \times Y = (T, +\infty) \times \Lambda_+$;
- the primitive f_L such that $df_L = e^t \alpha$ is a constant on $(-\infty, -T) \times \Lambda_-$ and on $(T, +\infty) \times \Lambda_+$.

Legendrians and Lagrangian cobordisms

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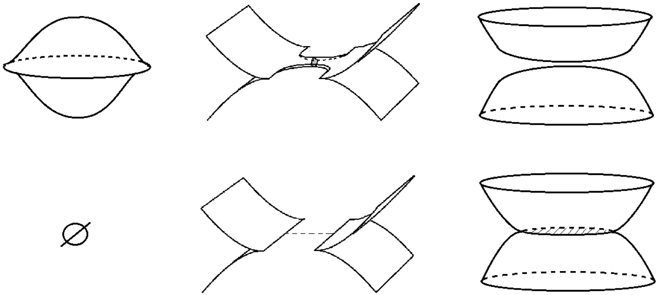

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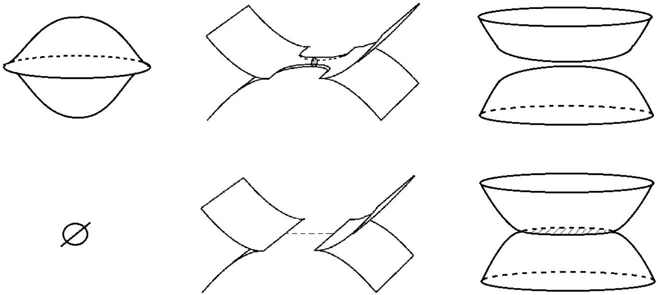

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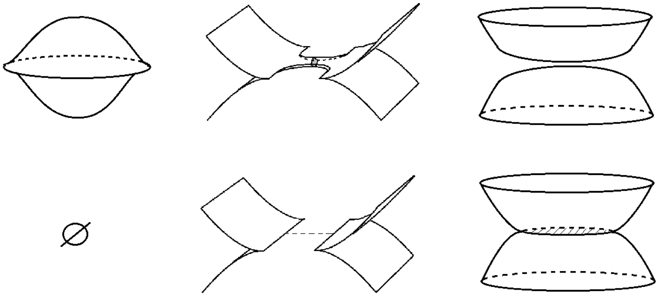
- There is a Lagrangian k-handle cobordism $0 \le k \le \dim \Lambda - 1$ (Dimitroglou Rizell).



Maps of Legendrian contact homology

- The Legendrian contact homology $\mathcal{A}(\Lambda)$ is a dg algebra generated by words of Reeb chords on the Legendrian Λ, whose differential counts pseudo-holomorphic curves in the symplectization $\mathbb{R}_t \times Y$ with boundary on $\mathbb{R}_t \times \Lambda$ (with one positive end and arbitrary negative ends).

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Microlocal theory of sheaves

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Definition

For a sheaf $\mathscr{F}$ on X, its singular support is the closure of the set of points $(x, \xi) \in T^{*,\infty}X$ such that for a small neighbourhood U of x and a smooth function f where $f(x) = 0, df(x) = \xi$,

$$\Gamma(U, \mathscr{F}) \to \Gamma(f^{-1}((-\infty, 0)), \mathscr{F})$$

is not an (quasi-)isomorphism.

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- For $\mathscr{F} \in Sh_\Lambda(M)$, at $(x, \xi) \in \Lambda$, the cone

$$\mathrm{Cone}(\Gamma(U, \mathscr{F}) \to \Gamma(f^{-1}((-\infty, 0)), \mathscr{F})) \neq 0$$

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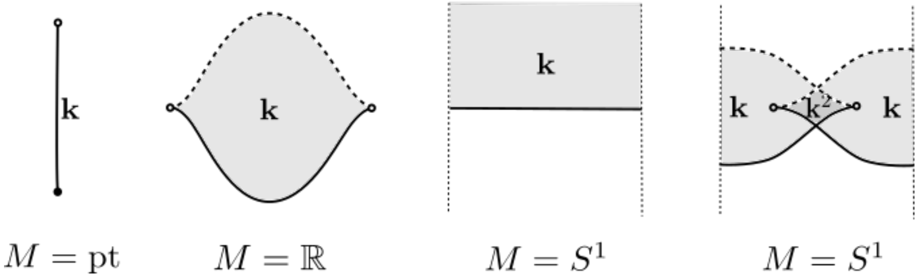
Theorem (Guillermou-Kashiwara-Schapira)

The category $Sh_\Lambda(M)$ of sheaves with $SS^\infty(\mathscr{F}) \subset \Lambda$ is invariant under Legendrian isotopies.

- One can view the 1-jet bundle $J^1(M) \cong T^{*,\infty}_{\tau>0}(M \times \mathbb{R}_t)$, and consider sheaves singularly supported on Legendrians $\Lambda \subset J^1(M)$.



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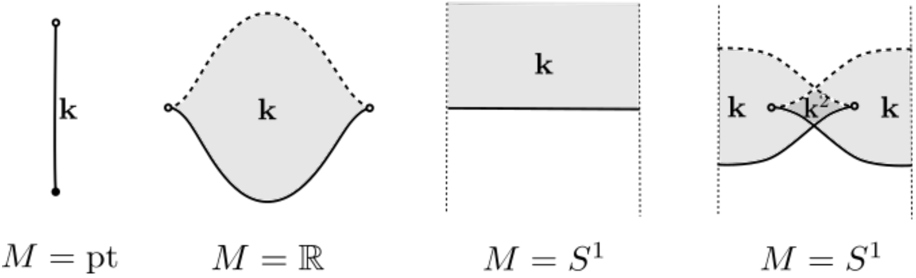
- Given a generating family $f : M \times F \to \mathbb{R}$ (with some control at infinity) for a Legendrian Λ, the fiberwise homology of $f^{-1}([0, +\infty))$ defines a sheaf with singular support on Λ (Viterbo).



Sheaves and Legendrian contact homology

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- For Legendrian links $\Lambda \subset J^1(\mathbb{R})$, Ng-Rutherford-Sivek-Shende-Zaslow shows that a category of "microlocal rank 1" sheaves with singular support on a Legendrian is equivalent to a $\mathcal{A}_\infty$-category of augmentations of the Legendrian contact homology with π_1-coefficients.

Sheaves and Legendrian contact homology

- Ganatra-Pardon-Shende shows the equivalence between the compact objects in (unbounded) category of sheaves with singular support Λ and the partially wrapped Fukaya category

$$Sh^c_\Lambda(M) \simeq \mathrm{Perf}\,\mathcal{W}(T^*M, \Lambda)^{\mathsf{op}}.$$

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- Legendrian surgery formula (Bourgeios-Ekholm-Eliashberg, Ekholm-Lekili, Ekholm and Asplund-Ekholm) shows the equivalence

$$\mathrm{Perf}\,\mathcal{W}(X, \Lambda) \simeq \mathrm{Perf}\,\mathcal{A}_{C_{-*}(\Omega_*\Lambda)}(\Lambda).$$

Main results

- Recall the expected Lagrangian cobordism maps between Legendrian contact homologies with loop space coefficients:

$$\Phi_L^* : \mathcal{A}_{C_{-*}(\Omega_*\Lambda_+)}(\Lambda_+) \to \mathcal{A}_{C_{-*}(\Omega_*\Lambda_-)}(\Lambda_-) \otimes_{C_{-*}(\Omega_*\Lambda_-)} C_{-*}(\Omega_* L),$$

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Theorem (L.)

Let $\Lambda_-, \Lambda_+ \subset S^*M$ and L an exact Lagrangian cobordism from Λ_- to Λ_+. Then there are functors between compact objects of sheaves

$$\Phi_L^* : Sh^c_{\Lambda_+}(M) \to Sh^c_{\Lambda_-}(M) \otimes_{Loc^c(\Lambda_-)} Loc^c(L).$$

and respectively, for proper modules of sheaves (these are constructible sheaves with perfect stalks) a fully faithful functor

$$\Phi_L : Sh^b_{\Lambda_-}(M) \times_{Loc^b(\Lambda_-)} Loc^b(L) \hookrightarrow Sh^b_{\Lambda_+}(M).$$

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Remark

The functor $Loc^b(L) \to Loc^b(\Lambda_-)$ *is the restriction. The functor* $Sh^b_{\Lambda_-}(M) \to Loc^b(\Lambda_-)$ *is given by computing microstalks (recall that microstalks are locally constant).*

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The compact objects of local systems $Loc^c(\Lambda_\pm)$ *and* $Loc^c(L)$ *are equivalent to perfect modules over chains on the based loop spaces* $C_{-*}(\Omega_*\Lambda_\pm)$ *and* $C_{-*}(\Omega_*L)$.

Main results

- Suppose the microstalks of $\mathscr{F}_-, \mathscr{G}_- \in Sh_{\Lambda_-}(M)$ define constant sheaves with stalks F and G. Gluing with constant sheaves on L, we can get sheaves $\mathscr{F}_+, \mathscr{G}_+ \in Sh_{\Lambda_+}(M)$.

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- Full faithfulness of the functor means the pullback/pushout square

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- The pullback/pushout square implies the relative pair/Mayor-Vitories/ Sabloff long exact sequences that one can deduce using Cthulhu complexes.

Applications

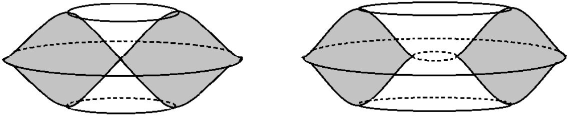

Figure: The Clifford Legendrian torus Λ_{Cliff} (on the left) and the unknotted Legendrian torus $\Lambda_{\mathrm{Unknot}}$ (on the right) in $J^1(\mathbb{R}^2)$.

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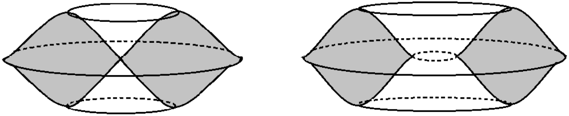

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Let $\Lambda_{g,k}$ be the Legendrian surface with genus g by taking k copies of Λ_{Cliff} and $g-k$ copies of Λ_{Unknot}. Then

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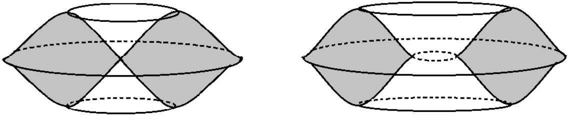

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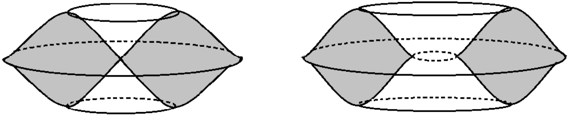

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2. *for any $k \geq 1, k' \geq 0$, there are cobordisms L from $\Lambda_{g,k}$ to $\Lambda_{g,k'}$ such that $\dim \operatorname{coker}(H^1(L) \to H^1(\Lambda_{g,k})) \geq 2$.*

Applications

- These Legendrian surfaces $\Lambda_{g,k}$ are considered Dimitroglou Rizell (ongoing project of Schrader, Shen & Zaslow also consider them). In particular, Dimitroglou Rizell proved that for $\Lambda_{g,k}$ is fillable if and only if $k = 0$. Hence there are no cobordisms from $\Lambda_{g,0}$ to $\Lambda_{g,k'}$ for $k' \geq 1$.

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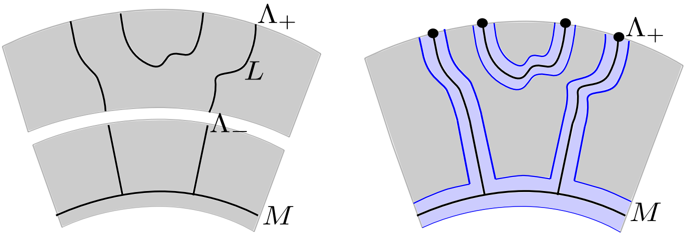
- Obstructions of H_1-surjective cobordisms are obtained using sheaves, by counting the number of microlocal rank 1 sheaves (presumably augmentations) over finite fields.
- Constructions of H_1-non-surjective cobordisms uses the result of Lagrangian caps by Eliashberg & Murphy.

Idea of the proof

- Consider the sheaf category $Sh_{\Lambda_\pm}(M)$ as categories on the Lagrangian skeleton $M \cup \mathbb{R}_{>0}\Lambda_\pm$, of the Weinstein sectors $(T^*M, \Lambda_\pm)$.



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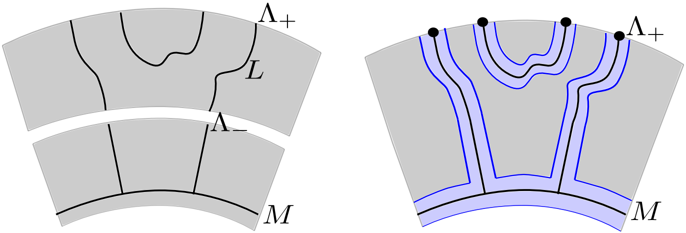

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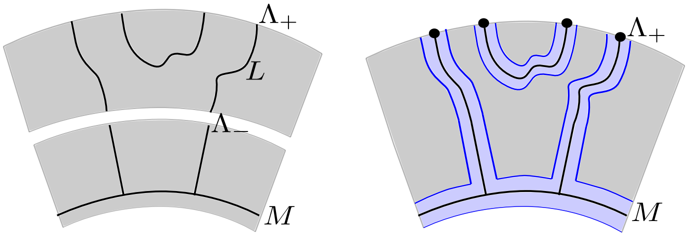
- Note that $L \hookrightarrow S^*M \times \mathbb{R}$, $(T^*M, \Lambda_-) \cup_{\Lambda_-} T^*L$ is embedded into (T^*M, Λ_+). The embedding functor maps $Sh_{\Lambda_-}(M) \times_{Loc(\Lambda_-)} Loc(L)$ to $Sh_{\Lambda_+}(M)$.



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- One technical difficulty for us is to show that composition of embeddings induce compositions of functors.

Alternative approach: sheaf quantization

- For Legendrians in $J^1(N)$, exact Lagrangian cobordisms between them are in $\mathbb{R} \times J^1(N)$ and have a Legendrian lifting into $(\mathbb{R} \times J^1(N)) \times \mathbb{R}$.

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- Another natural approach to build Lagrangian cobordism is to start from a sheaf near $N \times \{0\}$ in $Sh_{\Lambda_-}(N \times \mathbb{R})$, extend it to $Sh_{\widetilde{L}}(N \times \mathbb{R} \times \mathbb{R}_{>0})$, and then restrict to $N \times \{+\infty\}$ and get $Sh_{\Lambda_+}(N \times \mathbb{R})$.

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- For Legendrian knots in $J^1(\mathbb{R})$ or $J^1(S^1)$, Pan & Rutherford constructs the diagram (without adding loop space coefficients) for embedded cobordisms

$$\mathcal{A}(\Lambda_-) \xrightarrow{\sim} \mathcal{A}(\widetilde{L}) \leftarrow \mathcal{A}(\Lambda_+).$$

Alternative approach: sheaf quantization

- As one may expect, a necessary condition to extending the sheaf is that the microlocal monodromy data $Loc(\Lambda_-)$ extends to $Loc(L)$. Therefore, we hope to construct a functor

$$Sh_{\Lambda_-}(N \times \mathbb{R}) \times_{Loc(\Lambda_-)} Loc(L) \to Sh_{\widetilde{L}}(N \times \mathbb{R} \times \mathbb{R}_{>0})$$

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- We expect that on Legendrian contact homologies (with loop space coefficients), there is also a diagram

$$\mathcal{A}_{C_{-*}(\Omega_*\Lambda_-)}(\Lambda_-) \otimes_{C_{-*}(\Omega_*\Lambda_-)} C_{-*}(\Omega_* L) \xrightarrow{\sim} \mathcal{A}_{C_{-*}(\Omega_* L)}(\widetilde{L}) \leftarrow \mathcal{A}_{C_{-*}(\Omega_*\Lambda_+)}(\Lambda_+),$$

which leads to

$$\mathrm{Mod}\,\mathcal{A}_{C_{-*}(\Omega_*\Lambda_-)}(\Lambda_-) \times_{Loc(\Lambda_-)} Loc(L) \xleftarrow{\sim} \mathrm{Mod}\,\mathcal{A}_{C_{-*}(\Omega_* L)}(\widetilde{L}) \to \mathrm{Mod}\,\mathcal{A}_{C_{-*}(\Omega_*\Lambda_+)}(\Lambda_+).$$

Alternative approach: sheaf quantization

- We call the desired construction

$$Sh_{\Lambda_-}(N \times \mathbb{R}) \times_{Loc(\Lambda_-)} Loc(L) \to Sh_{\widetilde{L}}(N \times \mathbb{R} \times \mathbb{R}_{>0})$$

  a conditional sheaf quantization, as opposed to classical sheaf quantization of Lagrangians (with Legendrian lifts) of Guillermou and Jin–Treumann who construct

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- Work in progress will (hopefully) show that there is such a conditional sheaf quantization functor similar to Guillermou and Jin–Treumann's construction, and the cobordism functor obtained this way coincides the functor defined by embeddings of Lagrangian skeleta.

Thank you!